Mathematical Modeling in Finance

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CIRS

1. Introduction

The stock market has an important contribution in the rapidly growing world economy. The fluctuation in stock market can have a significant influence on persons and the entire economy. In the context of gathering money and capital formation, stock market is one of the best alternatives for many business houses and firms for further expansion and establishing a business venture (Fox et al., 2015). Stocks are the shares of a firm. The stock exchange is a legal framework in which a person or group can buy and sell such shares in a systematic manner. The stock market is the hub of both sellers and buyers of stock (Gomez et al., 2018). The development of the stock market is important in the economic growth of America. The stock market in the United States is in budding condition. The fundamental objective of New York Stock Exchange is to increase the liquidity and marketability of company securities by offering trading floor via market intermediaries and enabling and regulating trade securities.

The main objective of investors is to buy a stock is motivated by the desire for capital appreciation. Generally, the firms making more profit provide a greater return to the investors than those firms making less profit. The price of the share of the firms relies on their performance (Yuniningsih et al., 2015). There are various reasons that influence the overall trend of stock markets such as politico economic situation, natural disasters, poor-corporate governance, and differing policy of the governing company. In this regard, the return on the investment made by persons, corporate bodies in the stock market rely on the choice or decision of choosing appropriate firms to purchase stocks. In fact, the decision of choosing the most beneficial options in the stock market rely on how well a person informed in the stock analysis. This is the reason it is important to identify the statistical models and their analysis (Sen & Chaudhuri, 2016). These models assist to forecast the share price movement of stocks.

A competent stock market is considered to have such fundamental characteristics in which the price of shares should randomly change. The arbitrary fluctuation of shares prices causes the uniform distribution of market information. This inherent stochastic behavior of stock market makes the forecast of possible states of the market more sophisticated. However, there are various statistical models to study the phenomena of stock behavior. The Brownian motion model will predict the stock market using past information. The Geometric Brownian Motion will be applied to predict the Apple's stock price.

2. Background

The Brownian motion model of predicting stock behavior has its origins from Brownian motion concept. This Brownian motion phenomenon was discovered by a Scottish botanist named Robert Brown (Yao et al., 2017). The botanist was looking via a microscope at particles stuck in cavities in pollen grains when he discovered that the grains of pollen suspended in water had a quick oscillatory motion. Even though Brown published his observations, he was not capable to determine the mechanisms that facilitated this motion (Brown et al., 2017). Einstein published his classic paper where he clarified how the motion that Brown had observed was an outcome of the pollen being moved by personal water molecules (Kheifets et al., 2017). The concept is now referred to as Brownian motion. The precise description of Brownian motion is addressed later though Wiener presented a formal mathematical theory on the subject and therefore it is sometimes known as Wiener process (Le Gall, 2016).

The origins for the stochastic process as a mode of forecasting stock market behavior established by Bachelier. He created the first mathematical model of the price of a stock. He tested the model by utilizing it to price options and futures (Ikeda & Watanabe, 2014). Bachelier presumed the dynamics of stock market follows a Brownian motion with no time-value of funds.

Theorists Roberts, Kendall and Samuelson modified the Bachelier's model in order for the stock price to be followed by a log-normal distribution. Samuelson model was renamed the Geometric Brownian Motion (Belke et al, 2018).

Fama compared the behavior of stock price to the concept of a random walk (Barnes, 2016). The main assumption of a random walk is stock prices disclose information about the business (Mishra et al., 2015). In case the information available makes investors believe the firm will thrive, the confidence in the stock's value will soar and increase the demand for stocks leading to increasing in the stock's price. Conversely, if information available makes merchants believe the fortunes of the business will decrease, investor confidence in the stock will decrease. The demand for the company's stock will decrease leading to a reduction of the stock's prices. Because information arrives randomly, the prices of stocks have to change arbitrarily resulting to random walk concept. If the information flow is unhindered and is quickly replicated in the prices of stocks, then tomorrow's prices change will depict only the new of tomorrow and will be sovereign of the stock price changes today.

3. Method

Any variable whose value alters in an uncertain way is claimed to follow a stochastic process. Reddy & Clinton (2016) say that the concept of stochastic processes is significant in mathematical finance because it can be utilized to model many phenomena in which the quality or the factor differs continuously through time (Reddy & Clinton, 2016). Various processes are always modeled by a stochastic process and are a broad terminology for any assortment of random variables $[X(t)]$ relying on time t. Time might be discrete for instance, t=1,2 3, or continuous, $t \geq 0$.

The Brownian motion B *(t)* is utilized to capture the uncertainty in the future behavior of stochastic processes and has the subsequent features:

a. (Independence of increments) B*(t)*-B*(s)*, for t>s, is independent of the past.

b. (Continuity of paths) B (t), t≥ 0, are continuous functions of t.

c. (Normal increments) B*(t)*-B*(s)*, has Normal distribution with mean 0 and variance t-s, if $s = 0$ then B(t)-B(0)$\sim N(0, t)$.

Louis Bachelier used Brownian motion to model the prices of stocks. In a distinct form, the Bachelier model can be written as

$$dS_t = uS_t dt + \sigma S_t dW(t)$$

Where S*(t)* is the price of a stock, $B(t)$is the Brownian motion, u is the return on the price of a stock, σ is the volatility of the stock price. This equation is known as the arithmetic Brownian motion. The solution to Equation is $dS_t = uS_t dt + \sigma S_t dW(t)$ (Bae et al., 2015). Let us first comprehend this definition, normally, u is called the percentage drift and σ is known as percentage volatility. It is important to consider a Brownian motion course that satisfies this differential equation. The terminology $uS_t dt$ controls the 'trend' of this trajectory and the $\sigma S_t dW(t)$ regulates the random noise impact in the trajectory. It is vital to find a solution because it is a differential equation (Yang, 2015).

Using variable separation, the equation becomes

$$\frac{dS_t}{S_t} = udt + \sigma S_t W(t)$$

Then integrating both sides, we obtain

$$\int \frac{dS_t}{S_t} = \int \{udt + \sigma S_t W(t)\}dt$$

Because $\frac{dS_t}{S_t}$ links to the derivative of $ln(S_t)$, the preceding step will constitute the *Ito* calculus and resulting to the following equation.

$$ln(S_t) = \left(u - \frac{1}{2}\sigma^2\right)t + \sigma W(t)$$

Taking the exponential of both sides and plugging the first condition S_0 we obtain the solution. The analytical solution of this geometric Brownian solution is given by:

$$S_t = S_0\ exp\left(u - \frac{\sigma^2}{2}\right)t + \sigma W(t)$$

The process above is of solving a stochastic differential equation (SDE). A geometric Brownian motion is a stochastic differential equation (Dhesi & Ausloos, 2016). Hence, given a parameters σ and$u$, we are able to produce geometric Brownian motion throughout time interval. Before beginning the computer simulation, it is critical to understand that GBM solution can be expressed in the form:

$$S(t) = S_0 E^{X(t)}$$

Where

$$X(t) = \left(u - \frac{\sigma^2}{2}\right)t + \sigma W(t)$$

4. Prediction of the Apple's Stocks Prices Listed on New York Stock Exchange

I. Overview

Apple Stock is listed on the New York stock exchange where they the investors can buy and sell the stocks from an online platform. The New York Stock Exchange and the NASDAQ are the most prestigious stock exchanges across the world and most of the firms prefer or wish their stocks could be listed one of these stock exchanges (Yahoo Finance. 2015). Changes in regulatory issues have enabled exchanges to sell stocks that have been listed at other exchanges. Currently, the shares of the Apple Inc. shares are traded on NASDAQ, AmEx Stock Exchange, and the New York Stock Exchange (NYSE). The NYSE is the largest stock exchange market (Bessembinder et al, 2017).

II. Description of Apple Company

The company designs manufacture and sells media and communication devices and laptops. Moreover, it sells a diversity of associated software, accessories, and services, networking solutions and third-party digital applications and content. The product portfolio constitutes iPhone, iPad, Apple Watch, Mac, Apple TV. In addition, it vendors a portfolio of consumer and professional software applications, macOS, iOS, tvOS™ and watchOS operating systems (Apple, 2017).

The organization sells and delivers digital applications and content via the iTunes Store, Mac App Store, TV App Store, App Store, Apple Music and iBooks Store. The firm sells its products globally through its physical retail chains, e-stores, direct sales representatives, retailers, and wholesalers. It sells a diversity of third-party Apple-compatible products constituting accessories and application software via its online and retail chains. The firm sells to small and medium-sized business, enterprise, education and government consumers (Apple, 2017). Apple's financial year is the 52 or 53-week period that concludes on the final Saturday of September.

III. The volatility of Apple's Inc Stock Price

Apple's Inc. stock has encountered significant price volatility earlier and might linger to face this issue in the future. The firm, technology sector, and the stock market, in general, have encountered high levels of stock price and volume changes that have influenced stock prices in manners that might have been unassociated to this company's operating performance. The volatility of stock price in a certain period might cause the mean price where Apple Inc. repurchases its own stock to surpass the price of the stock at a specific time (Zucchi, 2015).

Apple Inc. considers its stock price should show expectations of high levels of profitability and future growth. Moreover, it believes its stock price should reflect anticipations that its cash dividend will linger at present values. It views that the cash dividend increases and that its present share repurchase program will be entirely consummated (Neto et al., 2017). The future dividends are posed for a declaration by the Company's Board of Directors, and the share repurchase program does not compel it to amass a specified number of shares. In case the firm fails to satisfy requirements for profitability, future growth, dividends, share repurchases, its stock price may decrease drastically. This could have a material adverse effect on worker retention and investor confidence (Ding et al., 2014).

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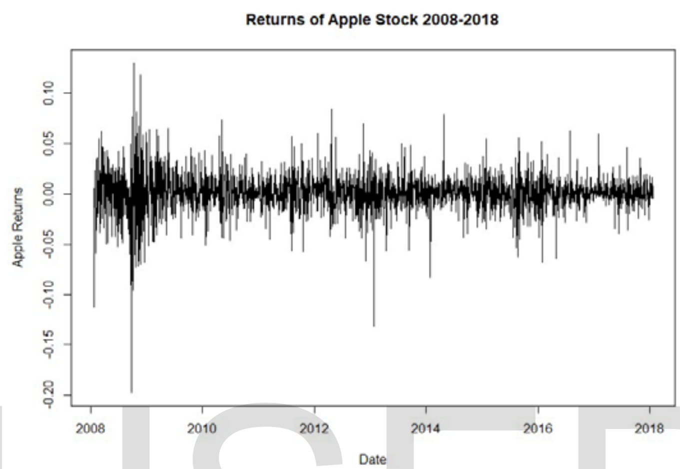
IV. The validity of the Geometric Brownian Motion

The historical closing prices of the Apple stock were compared to simulated prices by using basic statistical tests (Kyurkchiev, 2015). A time series of these prices of the Apple stock during the 2008 to 2018 period and the log returns of the series are illustrated in the graphs



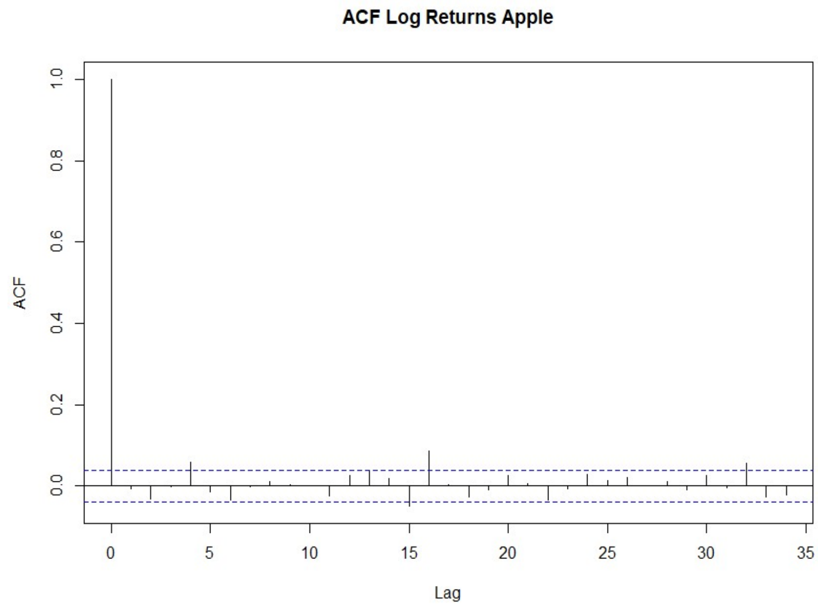
The 2008 financial crisis was associated with data that showed signs of volatility clustering with vast volatility. Here, $S(t_i)$ is the stock price at period t_i, The equation $r_i = \log\left(\frac{S(t_i)}{S(t_{i-1})}\right)$ is the log return at period t_i. Hence, the time series of stock prices can be addressed as $\{S(t_i)\}_{i=1}^{n}$ and the log returns series as $\{r_i\}_{i=1}^{n}$. Apple Inc. is a multinational consumer electronics firm founded in 1976. The company enjoyed total revenue of $61.1 billion in the last quarter of 2017. This signified a 16% increase from the previous quarter. The international sales accounted for 0.65 of the quarterly revenue (Apple, 2017).

In a standard time series, the stationary of the series is often presumed with constant variance and mean of the error terms. The latter are assumed to be independent and normally distributed.

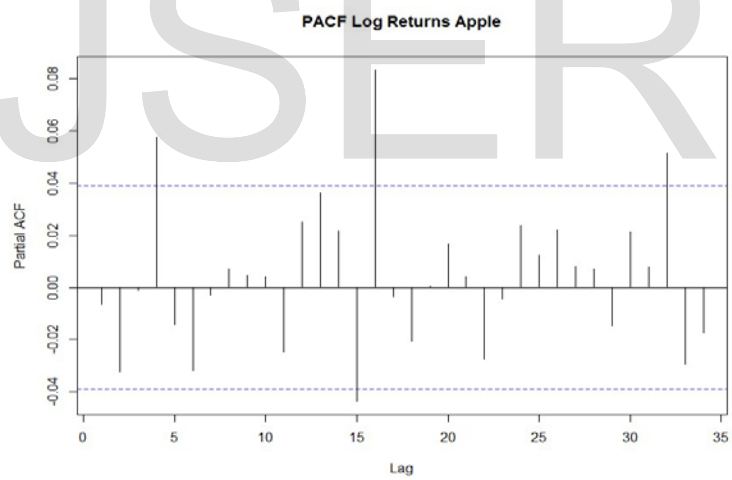
$$r_i = \log\left(\frac{S(t_i)}{S(t_{i-1})}\right) = \mu + \epsilon_i, \quad \epsilon_i \sim Normal(0, \sigma^2)$$

Some basic statistics of the log returns of the Apple stock price are presented in the table. The normal distribution is negatively skewed implying a left-skewed distribution (Explain). The normality presumption of the log returns is substantially rejected by the Jarque-Bera test. The dependence structure between the lag diminishes slowly because financial time series data shows evidence of "long memory" features. Hence, the Apple stock price data cannot be recognized as realizations of a stochastic process. The basic statistics of the log returns of Apple Inc. are shown below.

Sample size	**Max**	**Min**	**Mean**	**Standard Deviation**	**Skewness**	**Jarque-Bera Test**	**p-value**
2517	0.1301903	-0.1974729	0.0008274646	0.01950108	-0.5083595	7878.3	2.2e-16



(a) ACF of the Apple log returns series



(b) PACF of the Apple log returns series

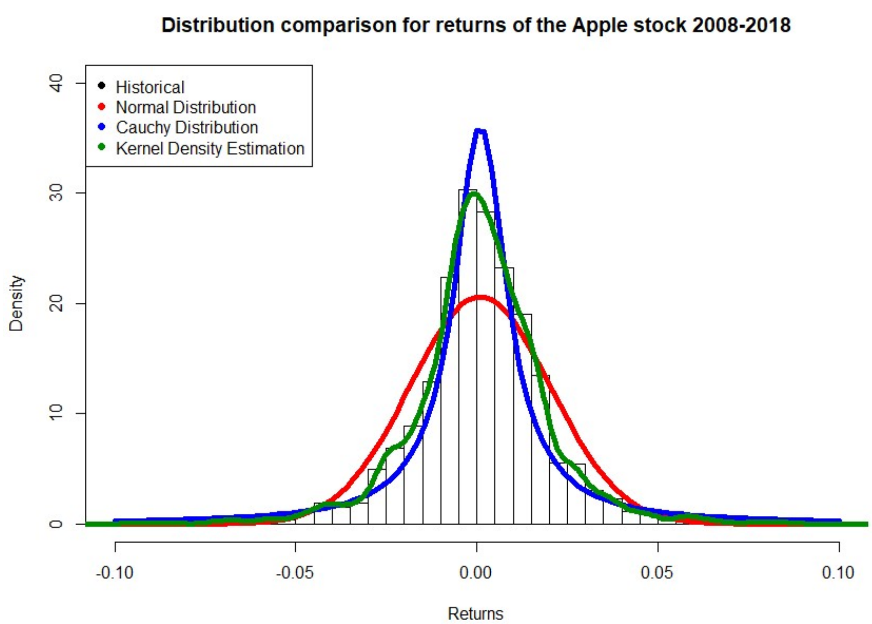
Autocorrelation and Partial autocorrelation of the Apple log returns

V. Distribution Assumption

A Geometric Brownian Motion assumes the logarithmic change of the price of a stock to be a normally distributed random variable based on:

$$r_i = \log\left(\frac{S(t_i)}{S(t_{i-1})}\right) = \mu + \epsilon_i, \quad \epsilon_i \sim Normal(0, \sigma^2)$$

This assumption can be tested against historical data as seen in the graph. The fitted normal distribution utilizes the whole sample period of one decade of closing prices of the apple stock. It approximates the expected variance and value of the logarithmic change of the stock prices which does not capture the real distribution. The distribution demonstrates signs of leptokurtosis. Hence, a modified distribution can be utilized to yield a better fit to the distribution of Apple returns (Dhesi et al., 2016). In the graph, the Cauchy distribution yields a better fit for the two data sets (Villari & Abdulla, 2017). However, the moments are not described. Using a non-parametric kernel density estimation results in an even better fit of the historical Apple returns.



The distribution of Apple returns imply a distribution that is slightly skewed. It has an appearance of more heavily-tailed than the Gaussian distribution which has zero skewness and a kurtosis of three.

5. Results and Discussion

I. Step Forward One Time Prediction

The GBM expressed in equation 9 is used to predict the price of Apple stock, however, the historical data used to estimate cannot be trusted. To estimate the drift and volatility, varying numbers of historical closings have been used for the investigation to see whether the use of longer or shorter time frames improves the predictions. By use of both the normality and Cauchy assumptions, distribution assumptions have been varied.

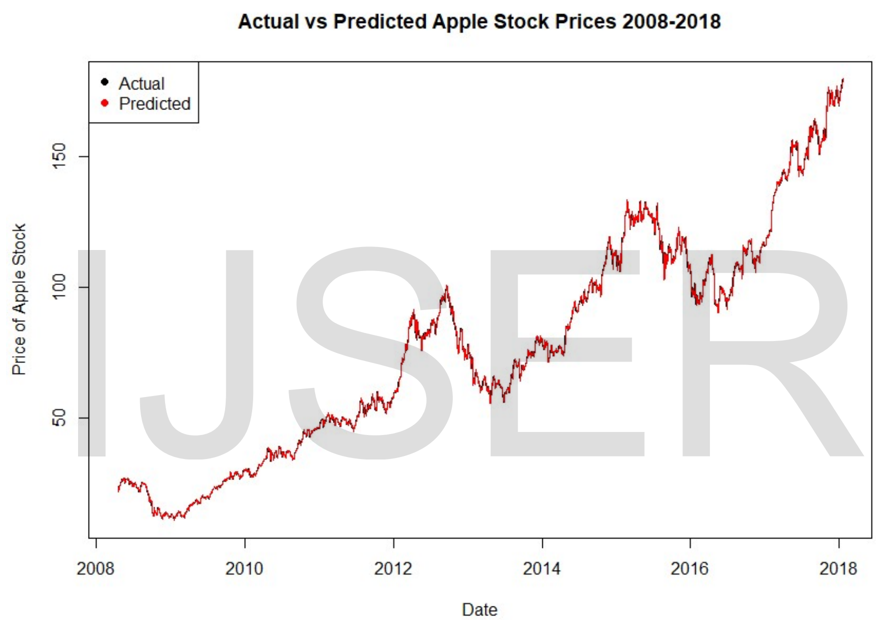
The results found in table below have been achieved through the use of Mean Square Error (MSE), and the share of the accurate top or bottom movements in the cost (p). The lowest MSE for the normality is achieved when using 60 days of historical closing prices while the highest probability is achieved when using 100 days by using the bootstrap estimate of the drift. Compared to the normality assumption, the Cauchy distribution provided lower MSE- values and the predictions were slightly above 50% of the time.

Sample size	MSENormal	pˆNormal	MSECauchy	pˆCauchy
20	1.68844	0.5252202	1.696666	0.5152122
30	1.676837	0.5188907	1.667149	0.5132637

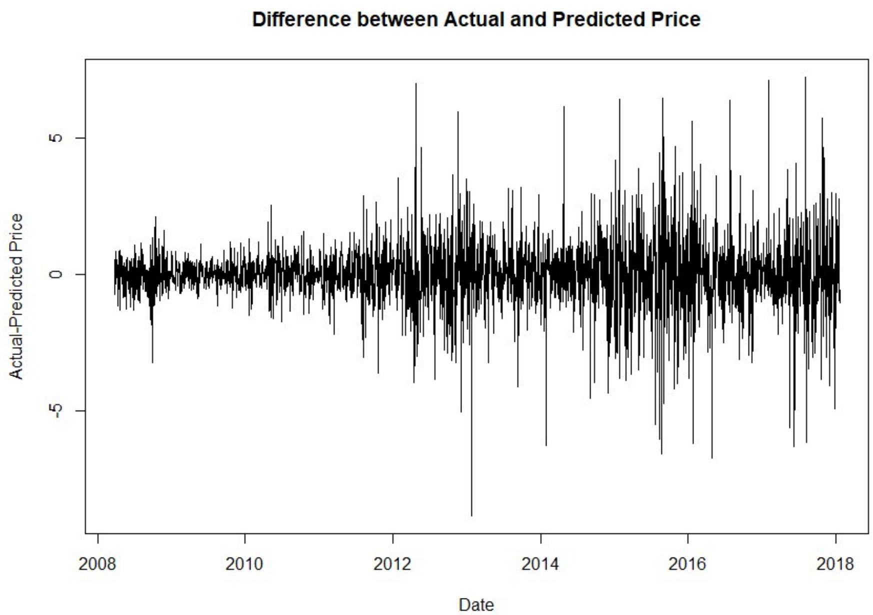
40	1.67217	0.5056497	1.666231	0.5092817
50	1.665432	0.5202593	1.65844	0.5149919
60	**1.66206**	**0.5174939**	**1.65229**	**0.5101709**
70	1.665484	0.5216503	1.657948	0.503268
80	1.66508	0.5307629	1.659285	0.5139459
90	1.673651	0.5201812	1.669046	0.5004119
100	1.676367	0.5326716	1.673352	0.5128205
200	1.714528	0.5181191	1.708248	0.5163934
300	1.782596	0.5081154	1.773526	0.5153291
400	1.859956	0.5080264	1.853068	0.5028329
500	1.941562	0.5208127	1.93728	0.5178394
600	2.018263	0.5218978	2.014919	0.5177268
700	2.105357	0.5192519	2.104171	0.5181518
800	2.202516	0.5157159	2.202301	0.5157159
900	2.299643	0.5173053	2.296151	0.5166873
1000	2.390633	0.5158103	2.386399	0.5158103

Using a varied time frame of the Apple stock to show a different outcome

In the figure below, using GBM with 60 days of historical data for the estimation of the drift and volatility, the actual figures of stock costs have been compared to the predicted stock prices. The results indicate that volatile periods yield a huge difference between actual and predicted price. The result is expected to be that way for GBM for it does not depend on the volatility, it depends on the drift.



(a) Apple stock returns during 2008-2018

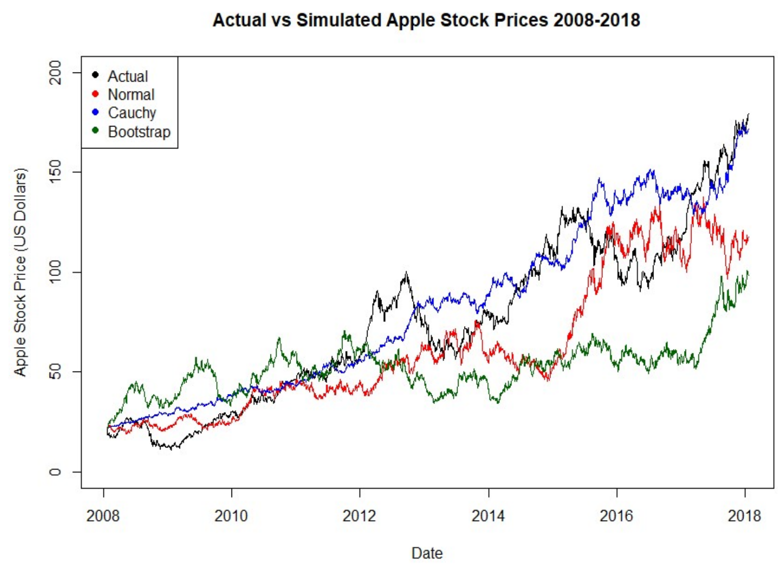


(b) Plot of differences

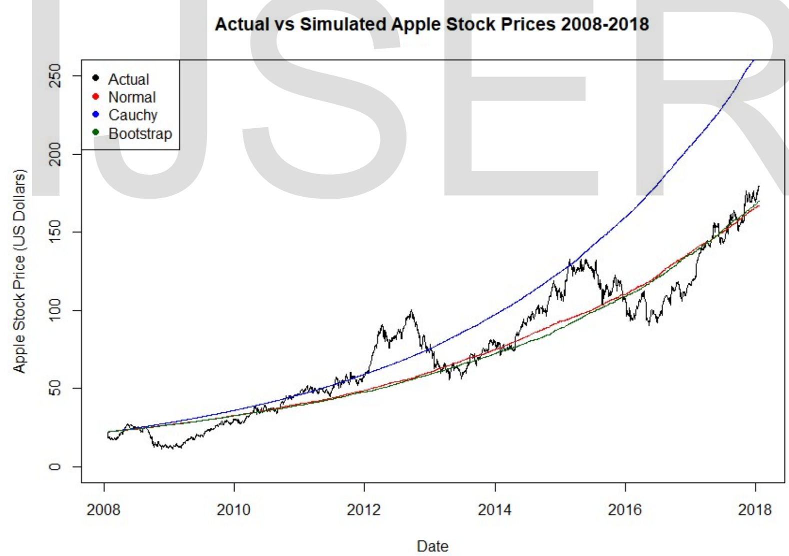
Actual vs Predicted Apple stock prices 2008-2018

II. A longer time Frame Prediction

Monte Carlo simulations have been made to make predictions of longer time frames of the Apple stock prices. It is not realistic to assume a longer time period of constant drift and volatility, hence the use of simulations that have different time frames being used for the estimation of the volatility and drift. Fig 9 is a sample of the above in use. In order to estimate drift and volatility for the normality assumption, the sample mean and standard deviation have been used. For Cauchy assumption, drift and volatility assumptions have been made by use of location and scale parameter. The figure below shows expected stock prices retrieved by 1000 Monte-Carlo simulations. During 2008-2018 the entire data set has been used to elaborate the drift and volatility used in the simulations.



Simulated Apple stock prices during 2008-2018



Expected Apple Stock prices during 2008-2018

a. Conclusion

The paper stochastically analyzes the stock market prices via a proposed lognormal model. The stock prices for one decade (from the New York Stock Exchange) were simulated.

This study examines the geometric Brownian model for predicting the stock price of Apple Inc. from 2008 to 2018. It provides two methods of testing the model validity. The initial method computes the correlation coefficient between the actual and stimulated prices of stocks. The earlier research have indicated that there is a weak association between these two factors. A negative correlation was evident during short run moments of stimulation, which becomes positive with elongated estimated horizons. The volatility in the stock market enables actual and stimulated stock price to have a negative correlation on a short period of time. On the contrary, stock prices stabilize to its average value, in the long run, resulting in a positive correlated between the real and stimulated prices of stocks. The correlation coefficient still epitomizers a weak association at best. One-time step forward predictions of the Apple stock price yielded the likelihoods of forecasting a correct down or up move of at least half. Hence the GBM with draft has some predictive power ingrained on a sample size at more than 2, 500 price observations. It is my hope that the Geometric Brownian motion model will be important to the Apple Inc. investors regarding predicting stock market behavior.

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